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The Inviscid Burgers Equation with Initial Value of Poissonian Type

A. Dermoune[1]

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We study the discontinuities (shocks) of the solution to the Burgers equation in the limit of vanishing viscosity (the inviscid limit) when the initial value is the opposite of the standard Poisson process p. We show that this solution is only defined for $t \in (0, 1)$. Let $T_0 = 0$ and T_n, $n \geqslant 1$, be the successive jumps of p. We prove that for all $M > 0$ the inviscid limit is characterized on the region $x \in (-\infty, M]$, $t \in (0, 1)$ by the increasing process $N(t) = \sup\{n \in \mathbb{N} \mid M + nt > T_n\}$ and the random set $I(x) = \{n \in \{0,..., N(t)\} \mid T_n - nt \leqslant x < T_{n+1} - nt\}$. The positions of shocks are given in a precise manner. We give the distribution of $N(t)$ and also the distribution of its first jump. We also prove similar results when the initial value is $u_\mu(y, 0) = -\mu p(y/\mu^2) + \mu^{-1} \max(y, 0)$, $\mu \in (0, 1)$.

KEY WORDS: Burgers equation; Kintchine–Pollaczek formula.

1. INTRODUCTION

The one-dimensional Burgers equation[1] without force has the form $\partial_t u + u\,\partial_x u = \nu\,\partial_x^2 u$, $x \in \mathbb{R}$, $t > 0$. Here ν is the viscosity. Introduce a potential function ψ, defined as $u = -\partial_x \psi$. The Hopf–Cole substitution $\psi = 2\nu \ln \theta$ shows that θ satisfies the heat equation $\partial_t \theta = \nu\,\partial_x^2 \theta$. Using this fact one can write down for the solution $u = u(x, t, \nu)$ the explicit expression

$$u(x, t, \nu) = \frac{\int_{-\infty}^{\infty} dy (x-y)/t \exp\{-F(x, y, t)/2\nu\}}{\int_{-\infty}^{\infty} dy \exp\{-F(x, y, t)/2\nu\}} \tag{1}$$

where

$$F(x, y, t) = \frac{(x-y)^2}{2t} + \int_0^y u(\eta, 0)\, d\eta$$

[1] Laboratoire de Statistique et Processus, Université Du Maine, 72017 Le Mans Cédex, France.

For the case $u(\eta, 0) = -p(\eta)$, p is the standard Poisson process on $\mathbb{R}_+$, (1) works only for $t \in (0, 1)$. In fact it is well known that $\lim_{s \to \infty} p(s)/s \to 1$ a.s., which implies $\lim_{y \to +\infty} 2 \int_0^y p(s)\, ds/y^2 = 1$ a.s., and for all $x \in \mathbb{R}$, $t \in (0, 1)$, $\lim_{y \to +\infty} 2F(x, y, t)/y^2 = 1 - t/t > 0$ a.s., which yields that (1) works for $t \in (0, 1)$. But for $u_\mu(\eta, 0) = -\mu p(\eta/\mu^2) + \mu^{-1} \max(\eta, 0)$ (1) works for $t \in (0, +\infty)$.

Burgers[1] and Hopf[3] discussed the behavior of solutions $u(x, t, v)$ as $v \to 0^+$ (inviscid limit[4, 5]) while the initial value $u(x, 0)$ is kept fixed. Let us formulate the following theorem by Hopf.[3]

Hopf's Theorem. For fixed x, t, the function $y \in \mathbb{R} \to F(x, y, t)$ attains its smallest value for one or several values of y, the smallest and the largest of which are denoted by $y_*(x, t)$ and $y^*(x, t)$, respectively. Then, for every x and t

$$\begin{aligned} \frac{x - y^*(x, t)}{t} &\leqslant \lim \inf_{v \to 0_+,\, \xi \to x,\, \tau \to t} u(\xi, \tau, v) \\ &\leqslant \lim \sup_{v \to 0_+,\, \xi \to x,\, \tau \to t} u(\xi, \tau, v) \leqslant \frac{x - y_*(x, t)}{t} \end{aligned}$$

In particular $\lim_{v \to 0_+,\, \xi \to x,\, \tau \to t} u(\xi, \tau, v) = x - y^*(x, t)/t = x - y_*(x, t)/t$ holds at every point, $x \in \mathbb{R}$, $t > 0$, where $y_* = y^*$.

This result is interpreted as follows. The inviscid limit $u(x, t, 0)$ is not well defined because it is multi-valued for those (x, t) where $y_*(x, t) \neq y^*(x, t)$. These points are interpreted as shocks, and $y^* - y_*/t$ is the size of the shock (see e.g.,[6] for a detailed study). The aim of this work is to characterize the locations of shocks in the cases where the initial value is equal to $u(y, 0) = -p(y)$ or $u_\mu(y, 0) = -\mu p(y/\mu^2) + \mu^{-1} \max(y, 0)$; $0 < \mu < 1$. The Burgers equation with Brownian motion as an initial value was proposed and studied numerically by She, Aurell and Frisch.[4] Ya. G. Sinai[5] obtained further rigorous quantitative results which have been numerically verified in ref. 4. It is well known that $u_\mu(y, 0)$ converges in law to the Brownian motion $B(y)$, so our study can be seen in a liberal sense as a macroscopic approach to the Brownian case.

Before giving the plan of the paper we need some notations. Let $M > 0$ and $N(t) = \sup\{n \in \mathbb{N} \mid M + nt > T_n\}$ where $T_0 = 0$ and $T_n, n \geqslant 1$ are the successive jumps of the Poisson process p. For the initial value $u(y, 0) = -p(y)$ we prove in Proposition 2.1 that for $x \in (-\infty, M]$, $t \in [0, 1)$ fixed the smallest value of $y \in \mathbb{R} \to F(x, y, t)$ is attained at several values among the points $x + mt$, where $m \in \{0,..., N(t)\}$ and we show that the inviscid limit $u(x, t, 0)$ is completely determined by the set $I(x) = \{n \in \{0,..., N(t)\} \mid T_n - nt \leqslant x < T_{n+1} - nt\}$. Theorem 2.1 gives the positions of shocks which belong to $(-\infty, M]$ at time $t \in (0, 1)$. More precisely, if we define for $S_0 = 0$, and $m \geqslant 1$

$$S_m = \sum_{i=1}^{m} \frac{m-i+1}{m} \tau_i - \frac{mt}{2} \qquad \text{where} \quad \tau_i = T_i - T_{i-1} \tag{2}$$

then we prove that shocks are located at $mS_m - nS_n/m - n$, for some $m \neq n$ belonging to $\{0,..., N(t)\}$. A precise characterization of these integers is also given.

In Proposition 3.1 we prove that there exists critical time J_1 such that for all $t \in [0, J_1)$, $N(t) = N(0)$. In Propositions 3.2, 3.3 we give the distributions of J_1 and $N(t)$ for all $t \in [0, 1)$. In Theorem 3.1 we prove that for $J = \min(J_1, \min\{\tau_{n+1} \mid 0 \leqslant n < N(0)\})$ the positions of shocks which belong to $(-\infty, M]$ at time $t \in [0, J)$ are $\{T_{n+1} - (2n + 1/2)\, t \mid 0 \leqslant n < N(0)\}$.

The study of the case when the initial value $u_\mu(y, 0) = -\mu p(y/\mu^2) + \max(y, 0)/\mu$ is similar to the latter case and we only mention this at the end of the Section 3. In the appendix we give an explicit description of the set $I(x)$.

Remark. If $u(y, 0) = p(y)$, or $u_\mu(y, 0) = \mu p(y/\mu^2) - \max(y, 0)/\mu$; μ is small, then one can easily see that there are no shocks at all.

2. THE CASE $u(y, 0) = -p(y)$

In this case $F(x, y, t)$ can be described explicitly as follows: if $y \in (-\infty, 0]$ then $F(x, y, t) = (x-y)^2/2t$. If $y \in (T_n, T_{n+1}]$ then

$$F(x, y, t) = \frac{(x-y)^2}{2t} - ny + \sum_{i=1}^{n} (n-i+1)\, \tau_i$$

and $\partial_y F(x, y, t) = t^{-1}(y - (x + tn))$.

For an interval I of $\mathbb{R}$, $y(x, t, I)$ denotes the y-coordinates of the minimum of the function $y \in I \to F(x, y, t)$, considered at fixed values of x and t. Then three cases can be distinguished:

$\mathscr{S}_1$: $(x + tn) \leqslant T_n$, in this case $y(x, t, [T_n, T_{n+1}]) = T_n$.

$\mathscr{S}_2$: $T_n \leqslant (x + nt) \leqslant T_{n+1}$, in this case $y(x, t, [T_n, T_{n+1}]) = x + nt$.

$\mathscr{S}_3$: $T_{n+1} \leqslant x + tn$, in this case $y(x, t, [T_n, T_{n+1}]) = T_{n+1}$. Now we can announce the following proposition.

Proposition 2.1. (1) Let $t \in [0, 1)$, for $M > 0$ the random variable $N(t) = \sup\{n \in \mathbb{N} \mid M + nt > T_n\}$ is a.s finite.

(2) Let $I(x) = \{n \in \{0,..., N(t)\} \mid T_n - nt \leqslant x < T_{n+1} - nt\}$, for all $x \leqslant M$,

$$y(x, t, \mathbb{R}) = y\left(x, t, \left[\frac{x - |x|}{2}, T_{N(t)}\right]\right) \subset \{x + nt, n \in I(x)\}$$

(3) The inviscid limit is given by $u(x, t, 0) = -p(y(x, t, \mathbb{R}))$ and its distribution is completely determined by the set $I(x)$.

Proof. The strong law of large numbers yields the finiteness of $N(t)$. Let us prove the rest of the proposition. If $x \leqslant T_n - nt$ then from $\mathscr{S}_1, \mathscr{S}_2, \mathscr{S}_3$ there exists an integer $m \leqslant n$ such that $F(x, x + mt, t) \leqslant F(x, T_n, t)$. If $x \geqslant T_{n+1} - nt$ then again from $\mathscr{S}_1, \mathscr{S}_2, \mathscr{S}_3$ there exists an integer $m \geqslant n$ such that $F(x, x + mt, t) \leqslant F(x, T_{n+1}, t)$. Now the statement of the proposition becomes obvious. A detailed description of the set $I(x)$ is given in the appendix.

Now let us study the position of shocks. From Proposition 2.1 we have to compare the quantities $F(x, x + mt, t)$ for $m \in I(x)$. Let S_m be the random variable defined by (2) then

$$F(x, x + mt, t) = m(S_m - x) \tag{3}$$

and $x + mt \in y(x, t, \mathbb{R})$ iff for all $n, j \in I(x)$ such that $n < m < j$,

$$\frac{mS_m - nS_n}{m - n} \leqslant x \leqslant \frac{mS_m - jS_j}{m - j} \tag{4}$$

Let $C(x)$ be the set of the integers $m \in I(x)$ satisfying (4), then we have $y(x, t, \mathbb{R}) = \{x + mt, m \in C(x)\}$, and x is a shock iff the set $C(x)$ contains at least two integers. The following theorem gives a precise characterization of the set $C(x)$.

Theorem 2.1. (1) If $m, n \in C(x)$ then

$$x = \frac{mS_m - nS_n}{m - n} \tag{5}$$

(2) For all t fixed the set $C(x)$ contains at the most two integers.

(3) The integers $n < m$ belong to $C(x)$ iff the three following conditions are satisfied:

(i) $T_m - mt \leqslant x \leqslant T_{m+1} - mt$, $T_n - nt \leqslant x \leqslant T_{n+1} - nt$,

(ii) $\min_{n<j,\, j\in I(x)} jS_j - nS_n/j - n = mS_m - nS_n/m - n = \max_{j<m,\, j\in I(x)} jS_j - mS_m/j - m = x$,

(iii) $\max_{j<n,\, j\in I(x)} jS_j - nS_n/j - n \leqslant x \leqslant \min_{m<j,\, j\in I(x)} mS_m - jS_j/m - j$.

Proof. (1) If $m, n \in C(x)$ then $F(x, x+nt, t) = F(x, x+mt, t)$, and from (3) we have the formula (5).

(2) Suppose that $m, n, j \in C(x)$, from (5) we have $mS_m - nS_n/m - n = mS_m - jS_j/m - j$, which implies that t is not deterministic.

(3) Let $n < m \in C(x)$, then $m, n \in I(x)$, which is equivalent to the condition (i). From (4), (5) we have the conditions (ii) and (iii). The proof of the part "only if" is easy.

3. THE LOCATIONS OF SHOCKS WHEN THE TIME IS NOT EXCEEDING CRITICAL VALUES

The following proposition gives the variation of $N(t)$ with respect to the time t.

Proposition 3.1. Let us define $J_1 = \min_{k>N(0)} T_k - M/k$. Then $t \in (0, 1) \to N(t)$ is an increasing and $N(0) = N(t)$ for all $t \in [0, J_1)$.

Proof. It is easy to show that $t \to N(t)$ is increasing. The integer $N(t)$ is the smallest integer satisfying $t < \min_{k>N(t)} T_k - M/k$, so if $t \in [0, J_1)$ then $N(t) \leqslant N(0)$. Now from the monotonicity of $t \to N(t)$ the result $N(0) = N(t)$, $\forall t \in [0, J_1)$ becomes obvious.

The following result gives for $t \in [0, 1)$ the law of $N(t)$.

Proposition 3.2. For $t \in [0, 1)$ let us define

$$\beta(t) = \left[1 - (1-t) \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} \{(j+1)!\}^{-1} (jt)^{j+1} \times \left\{ 1 - \left(1 - \frac{k}{j}\right)^{j+1} \right\} \exp(-jt) \right]$$

then we have for all $n \in \mathbb{N}$

$$P(N(t) = n) = \frac{(M+nt)^n}{n!} \exp(-(n+1)\, t - M)\, \beta(t)$$

In particular $N(0)$ has the Poisson distribution with parameter M.

Proof. Let $a_n(t)$ be the real $M+nt$. We have

$$[N(t)=n]=[T_n \leqslant a_n(t)] \cap \bigcap_{m>n}^{\infty} [T_m > a_m(t)]$$

$$=[T_n \leqslant a_n(t) \leqslant a_{n+1}(t) \leqslant T_{n+1}]$$

$$\cap \bigcap_{m>n+1}^{\infty} [T_m - T_{n+1} - (m-n-1)\, t \geqslant a_{n+1}(t) - T_{n+1}]$$

Let us define for $n \in \mathbb{N}$ and $t \in (0, 1)$ the random walk $R_m^n(t)$ by $\sum_{i=1}^{m} (\tau_i^n - t)$, where $\tau_1^n, \tau_2^n, \ldots$ are mutually independent random variables with common exponential distribution with parameter 1 and independent of $T_1, \ldots, T_{n+1}$. From the independent of $(T_1, \ldots, T_{n+1})$ and $(T_m - T_{n+1}, m > n+1)$, the probability $P(N(t)=n)$ is the product of the two following probabilities: $P(T_n \leqslant a_n(t) \leqslant a_{n+1}(t) \leqslant T_{n+1})$ and $P(\min_{k \geqslant 1} R_k^n(t) \geqslant a_{n+1}(t) - T_{n+1} \mid T_n \leqslant a_n(t) \leqslant a_{n+1}(t) \leqslant T_{n+1})$.

The probability

$$P(\min_{k \geqslant 1} R_k^n(t) \geqslant a_{n+1}(t) - T_{n+1} \mid T_n \leqslant a_n(t) \leqslant a_{n+1}(t) \leqslant T_{n+1})$$

$$= 1 - P(\min_{k \geqslant 1} R_k^n(t) < a_{n+1}(t) - T_{n+1} \mid T_n \leqslant a_n(t) \leqslant a_{n+1}(t) \leqslant T_{n+1}).$$

From the Khintchine–Pollaczek formula[2] (p. 392), we have for all $x<0$ that $P(\min_{m \geqslant 1} R_m^n(t) \leqslant x) = (1-t) \sum_{k=0}^{\infty} d\rho^{*k}(-\infty, x]$, where

$$\rho(x) = \begin{cases} 0 & \text{for } \ x \leqslant -t \\ x+t & \text{for } \ -t \leqslant x \leqslant 0 \end{cases}$$

From the Laplace transform we can see that for $n \in \mathbb{N}$, $-(n+1)\, t \leqslant x \leqslant -nt$, $\sum_{k=0}^{\infty} d\rho^{*k}(-\infty, x] = \sum_{j=n+1}^{\infty} (x+jt)^j/j! \exp(-x-jt)$. Using that and using the law of (T_n, T_{n+1}) we obtain the result.

The following proposition gives a simplified form of $\beta(s)$ and the distribution of J_1.

Proposition 3.3. From the notations of Proposition 3.2 we have for $s \in [0, 1)$:

(1) $\beta(s) = \{\sum_{n=0}^{\infty} (ns)^n/n!\, e^{-ns}\}^{-1}$.

(2) $P(J_1 \geqslant s) = \exp(-M - s + Me^{-s})\, \beta(s)$.

Proof. From Proposition 3.2 we have for all $s \in [0, 1)$ and for all $M > 0$, $n \in \mathbb{N}$ that $\sum_{n=0}^{\infty} (M+ns)^n/n! \exp(-M-ns)\, \beta(s) = 1$. By tending $M \to 0^+$ we have the result (1). Let us prove (2).

$$
\begin{aligned}
P(J_1 \geqslant s) &= P\left(\min_{k > N(0)} \frac{T_k - M}{k} \geqslant s\right) \\
&= \sum_{j=0}^{\infty} P\left(\min_{k > j} \frac{T_k - M}{k} \geqslant s, N(0) = j\right) \\
&= \sum_{j=0}^{\infty} P(\min_{k > j+1} (T_k - T_{j+1} - (k-j-1)\, s) \\
&\geqslant a_{j+1}(s) - T_{j+1}, T_j < M < a_{j+1}(s) < T_{j+1})
\end{aligned}
$$

Using the independent of $(T_k - T_{j+1} - (k-(j+1))\, s, k > j+1)$ and $T_1,..., T_{j+1}$ we have

$$
P(J_1 \geqslant s) = \sum_{j=0}^{\infty} P(T_j < M < a_{j+1}(s) < T_{j+1})
$$

$$
P(\min_{k \geqslant 1} R_k^j(s) \geqslant a_{j+1}(s) - T_{j+1} \mid T_j < M < a_{j+1}(s) < T_{j+1})
$$

Now from the Khintchine–Pollaczek formula we have the result.

The following theorem gives explicitly the position of shocks when the time t is not exceeding J_1.

Theorem 3.1. If $t < J = \min(J_1, \min\{\tau_{n+1} \mid 0 \leqslant n < N(0)\})$ then the position of shocks belonging to $(-\infty, M]$ are given by

$$
T_{n+1} - \frac{2n+1}{2} t, 0 \leqslant n < N(0)
$$

Proof. The initial value $u(x, 0) = -p(x)$ is such that $u(T_{n+1}^-, 0) > u(T_{n+1}^+, 0)$ for all $n \in \mathbb{N}$. Then[3] T_{n+1} is the starting point of a discontinuity line $s_{n+1}(t)$ for the inviscid limit $u(x, t, 0)$. This line satisfies the Rankine–Hugoniot condition

$$
s'_{n+1}(t) = -\frac{2n+1}{2}, \qquad s_{n+1}(0) = T_{n+1}
$$

From that we obtain $s_{n+1}(t) = T_{n+1} - \frac{2n+1}{2} t$. For $t < \min\{\tau_{n+1} \mid 0 \leqslant n < N(0)\}$ the discontinuity lines $s_{n+1}, 0 \leqslant n < N(0)$ cannot intersect. From Propositions 2.1, 3.1 we conclude that the discontinuity lines in the region $x \in (-\infty, M]$, $t \in [0, J)$ are the segments $s_{n+1}, 0 \leqslant n < N(0)$. Which achieves the proof.

Now we show how we can derive the study of the case $u_\mu(y,0)$. If $y \in (-\infty, 0]$ then we have $F(x,y,t) = (x-y)^2/2t$. If $y \in (\mu^2 T_n, \mu^2 T_{n+1}]$ then

$$F(x, y, t) = \frac{(x-y)^2}{2t} + \frac{y^2}{2\mu} - n\mu y + \mu^3 \sum_{i=1}^{n} (n-i+1)\,\tau_i$$

and $\partial_y F(x,y,t) = \frac{\mu+t}{\mu t}\left(y - \frac{\mu(x+n\mu t)}{\mu+t}\right)$. We distinguish three cases: $(x+n\mu t) \leqslant \mu(\mu+t)\,T_n$ implies that

$$y(x, t, [\mu^2 T_n, \mu^2 T_{n+1}]) = \mu^2 T_n$$

$\mu(\mu+t)\,T_n \leqslant (x+n\mu t) \leqslant \mu(\mu+t)\,T_{n+1}$ implies that

$$y(x, t, [\mu^2 T_n, \mu^2 T_{n+1}]) = \frac{\mu(x+n\mu t)}{\mu+t}$$

and $\mu(\mu+t)\,T_{n+1} \leqslant (x+n\mu t)$ implies that

$$y(x, t, [\mu^2 T_n, \mu^2 T_{n+1}]) = \mu^2 T_{n+1}$$

For $M>0$ and $t \geqslant 0$ let us define the random variable $N_\mu(t) = \sup\{n \in \mathbb{N} \mid (M+n\mu t) > \mu(\mu+t)\,T_n\}$. We have a similar statement to Proposition 2.1 which gives that for all $x \leqslant M$,

$$y(x, t, \mathbb{R}) = y\left(x, t, \left[\frac{x-|x|}{2}, \mu^2 T_{N(\mu)}\right]\right) \subset \left\{\frac{\mu(x+n\mu t)}{\mu+t} \,\middle|\, n \leqslant N(\mu)\right\}$$

Let $m \leqslant N_\mu(t)$ such that $\mu(x+m\mu t)/\mu+t \in y(x,t,\mathbb{R})$, then the quantity $F(x, \mu(x+m\mu t)/\mu+t, t)$ is equal to

$$\frac{x^2+tm^2\mu^3}{2(\mu+t)} - m\mu^2\,\frac{(x+m\mu t)}{\mu+t} + \mu^3 \sum_{i=1}^{m} (m-i+1)\,\tau_i \tag{6}$$

Let us denote $S_m(\mu) = \mu(\mu+t)\sum_{i=1}^{m}(m-i+1)\,\tau_i - m^2\mu t/2$. A point x is a position of the shock if there exist $k, m \leqslant N(\mu)$ such that $\mu(x+m\mu t)/\mu+t$ and $\mu(x+k\mu t)/\mu+t$ are the absolute minima of the map $y \in \mathbb{R} \to F(x,y,t)$. From (6) the shock points have necessary the form $x(\mu) = \dfrac{S_m(\mu) - S_k(\mu)}{m-k}$. To characterize them, we can repeat the same arguments as before, $I_\mu(x) = \{n \in \{0,\ldots, N_\mu(t)\} \mid \mu(\mu+t)\,T_n \leqslant (x+n\mu t) \leqslant \mu(\mu+t)\,T_{n+1}\}$ and $S_n(\mu)$ take the place respectively of $I(x)$ and S_n.

APPENDIX

Now we give a detailed description of the random set $I(x)$. From the Proposition 2.1 the study of $I(x)$ for $x \leqslant M$ depends on the position of x w.r.t. $Y_k = T_k - kt$, and $Z_k = T_{k+1} - kt$; $k \leqslant N(t)$. The sequences $(Y_0, ..., Y_{N(t)})$, $(Z_0, ..., Z_{N(t)})$ are not ordered, so let us order them: $Y_{(0)} < \cdots < Y_{(N(t))}$, and $Z_{(0)} < \cdots < Z_{(N(t))}$. We denote by $n(k, Y)$ (respectively $n(k, Z)$) the integer n such that $0 \leqslant n \leqslant N(t)$; and $Y_n = Y_{(k)}$ (respectively $Z_n = Z_{(k)}$). We have a.s. two bijective maps from $\{0, ..., N(t)\}$ into itself, namely, $k \in \{0, ..., N(t)\} \to n(k, Y) \in \{0, ..., N(t)\}$ and $k \in \{0, ..., N(t)\} \to n(k, Z) \in \{0, ..., N(t)\}$.

Now we order the sequence $(Y_{n(0, Y)}, ..., Y_{n(N(t), Y)}, Z_{n(0, Z)}, ..., Z_{n(N(t), Z)})$. Define for $0 \leqslant k \leqslant N(t)$ the integer $a(k) = \sup\{0 \leqslant j \leqslant N(t) \mid Y_{n(j, Y)} < Z_{n(k, Z)}\}$. It is easy to see that the sequence $a(k)$, $0 \leqslant k \leqslant N(t)$ is increasing but not strictly increasing. Let $r_0, ..., r_p$ be the integers such that $\sum_{i=0}^{p} r_i = N+1$, and $a(0) = \cdots = a(r_0) < a(r_0+1) = \cdots = a(r_0+r_1) < a(r_0+r_1+1) \cdots < a(r_0 + \cdots + r_{p-1}+1) = \cdots = a(N(t))$. Then we have

$$
\begin{aligned}
&Y_{(0)} < \cdots < Y_{(a(0))} < Z_{(0)} < \cdots < Z_{(r_0)} \\
&\quad < Y_{(a(0)+1)} < \cdots < Y_{(a(r_0+1))} < Z_{(r_0+1)} < \cdots \\
&\quad < Y_{(a(r_0+\cdots r_{i-1}+1))} < Z_{(r_0+\cdots+r_{i-1}+1)} < \cdots < Z_{(N(t))}
\end{aligned}
$$

From that and from the inequality $Y_k \leqslant Z_k$ for all k, we have the relations $r_0 \leqslant a(0)$, $r_0 + \cdots + r_i \leqslant a(r_0 + \cdots + r_{i-1} + 1)$. This implies $\{n(0, Z), ..., n(r_0, Z)\} \subset \{n(0, Y), ..., n(a(0), Y)\}$, and for $i \geqslant 1$ $\{n(0, Z), ..., n(r_0 + r_1 + \cdots + r_i, Z)\} \subset \{n(0, Y), ..., n(a(r_0 + \cdots + r_{i-1} + 1), Y)\}$.

The following descriptions follow from the definition of $I(x)$ and from the notations above.

(B_1) If $Y_{n(i, Y)} \leqslant x \leqslant Y_{n(i+1, Y)}$; $0 \leqslant i \leqslant a(0) - 1$, then

$$I(x) = \{n(0, Y), ..., n(i, Y)\}$$

(B_2) If $Y_{n(a(0), Y)} \leqslant x \leqslant Z_{n(0, Z)}$ then

$$I(x) = \{n(0, Y), ..., n(a(0), Y)\}$$

(B_3) If $Z_{n(i, Z)} \leqslant x \leqslant Z_{n(i+1, Z)}$; $0 \leqslant i < r_0$, then

$$I(x) = \{n(0, Y), ..., n(a(0), Y)\} \cap \{n(i+1, Z), ..., n(N, Z)\}$$

(B_4) If $Z_{n(r_0, Z)} \leqslant x \leqslant Y_{n(a(0)+1, Y)}$ then

$$I(x) = \{n(0, Y), ..., n(a(0), Y)\} \cap \{n(r_0 + 1, Z), ..., n(N, Z)\}$$

(B_5) If $Y_{n(i,Y)} \leqslant x \leqslant Y_{n(i+1,Y)}$; $a(0)+1 \leqslant i < a(r_0+1)$ then

$$I(x) = \{n(0, Y),..., n(i, Y)\} \cap \{n(r_0+1, Z),..., n(N, Z)\}$$

(B_6) If $Y_{n(a(r_0+1),Y)} \leqslant x \leqslant Z_{n(r_0+1,Z)}$ then

$$I(x) = \{n(0, Y),..., n(a(r_0+1), Y)\} \cap \{n(r_0+1, Z),..., n(N, Z)\}$$

(B_7) If $Y_{n(i,Y)} \leqslant x \leqslant Y_{n(i+1,Y)}$; $a(r_0+\cdots+r_{j-1}+1)+1 \leqslant i < a(r_0+\cdots+r_j+1)$; $1 \leqslant j \leqslant p-1$, then

$$I(x) = \{n(0, Y),..., n(i, Y)\} \cap \{n(r_0+\cdots+r_j+1, Z),..., n(N, Z)\}$$

(B_8) If $Z_{n(i,Z)} \leqslant x \leqslant Z_{n(i+1,Z)}$; $r_0+\cdots+r_j+1 \leqslant i < r_0+\cdots+r_{j+1}$; $0 \leqslant j \leqslant p-1$, then

$$I(x) = \{n(0, Y),..., n(a(r_0+\cdots+r_j+1), Y)\} \cap \{n(i+1, Z),..., n(N, Z)\}$$

(B_9) If $Z_{n(r_0+\cdots+r_j,Z)} \leqslant x \leqslant Y_{n(a(r_0+\cdots+r_{j-1}+1)+1,Y)}$; $1 \leqslant j \leqslant p-1$, then $I(x)$ is equal to

$$\{n(0, Y),..., n(a(r_0+\cdots+r_{j-1}+1), Y)\}$$
$$\cap \{n(r_0+\cdots+r_j+1, Z),..., n(N, Z)\}$$

(B_{10}) If $Y_{n(a(r_0+\cdots+r_j+1),Y)} \leqslant x \leqslant Z_{n(r_0+\cdots+r_j+1,Z)}$; $0 \leqslant j \leqslant p-1$, then $I(x)$ is equal to

$$\{n(0, Y),..., n(a(r_0+\cdots+r_j+1), Y)\} \cap \{n(r_0+\cdots+r_j+1, Z),..., n(N, Z)\}$$

Remark that if x lies in $[Y_{n(i,Y)}, Y_{n(i+1,Y)}]$, $[Y_{n(a(k),Y)}, Z_{n(k,Z)}]$, $[Z_{n(i,Z)}, Z_{n(i+1,Z)}]$ or lies in $[Z_{n(r_0+\cdots+r_j,Z)}, Y_{n(a(r_0+\cdots+r_{j-1}+1),Y)}]$ then the set $I(x)$ depends only on these intervals.

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